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THE USE OF CASE'S METHOD TO SOLVE THE LINEARIZED BGK EQUATIONS FOR THE TEMPERATURE-JUMP PROBLEM*

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The temperature-jump problem in a rarefied gas occupying a half-space with given temperature gradient at infinity is considered. Case's method is used to find the eigenvectors and eigenvalues of the appropriate transport operators. A completeness theorem for the family of eigenvectors is proved by solving the vector Riemann-Hilbert problem with matrix coefficients. The matrix that puts the boundary-value problem coefficient into diagonal form is analytic in the complex plane with two cuts joining two pairs of branch pints. This necessitates solving an additional boundary-value problem on the cuts, by means of which a fundamental matrix is constructed. A solution of the problem is constructed using this matrix, and, as an application, an exact formula is obtained for calculating the temperature jump by quadratures.

Case's method /1/ was first applied /2/ to kinetic theory, and an analytic solution of the temperature-jump problem for the Boltzmann BGK-equations was presented in /3/. However, the vectors $\tilde{u}$ and $\tilde{w}$ in /3/ have different limiting values above and below the point z_0 on the positive half-axis $\mathbb{R}_+$. The solution in /3/ is therefore wrong.

Based on the results of /3/ a canonical matrix was constructed /4/ for the Riemann-Hilbert boundary-value problem. An approximate but (according to its authors) "high-precision" numerical value for the magnitude of the temperature jump is given in /5/.

A bibliography for this problem is given in /3-5/, and a theory of exact solutions of linearized vector kinetic equations of the form

$$\mu \frac{\partial}{\partial x} \Psi(x, \mu) + \Sigma \Psi(x, \mu) = \int_{-\infty}^{\infty} \exp(-\mu'^2) K(\mu, \mu') \Psi(x, \mu') \, d\mu'$$

$$\Sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_N), \quad \Psi(x, \mu) = (\psi_1(x, \mu), \ldots, \psi_N(x, \mu))^T$$

where Σ is a diagonal transport matrix, $K(\mu, \mu')$ is an $(N \times N)$ matrix and $\Psi(x, \mu)$ is the unknown vector, is constructed in /6/.

Suppose that a rarefied monatomic gas occupies the half-space $x > 0$, and that far from the $x = 0$ plane a stationary temperature field

$$T(x) = T_0 (1 + kx) \quad (x \to \infty)$$

is maintained in the gas.

Omitting minor details to be found in /3-5/, we reduce the temperature jump problem to the solution of a vector equation with boundary conditions

$$\mu \frac{\partial}{\partial x} Z(x, \mu) + Z(x, \mu) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-\mu'^2) K(\mu') Z(x, \mu') \, d\mu' \tag{1}$$

$$Z(0, \mu) = A\mu \begin{Vmatrix} -1 \\ 1/\gamma \end{Vmatrix} + \varepsilon_n \begin{Vmatrix} 1 \\ 0 \end{Vmatrix} \ (\mu > 0), \quad Z(\infty, \mu) = \varepsilon_T \begin{Vmatrix} -1 \\ 1/\gamma \end{Vmatrix} \tag{2}$$

Here

$$K(\mu) = \begin{Vmatrix} 1 & \varkappa \\ \varkappa & \gamma^2 + \varkappa^2 \end{Vmatrix}, \quad \varkappa = \gamma\left(\mu^2 - \frac{1}{2}\right), \quad \gamma^2 = \frac{2}{3}$$

$$A = \frac{3lk}{\sqrt{\pi}}, \quad \varepsilon_T = \frac{T_0 - T_w}{T_w}$$

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(where ε_T is the unknown temperature jump, T_w is the temperature of the wall, and $Z(x, \mu)$ is a column vector).

If we seek a solution of (1) in the form

$$Z(x, \mu) = \exp(-x/\eta) F(\eta, \mu) \tag{3}$$

we obtain, after lengthy algebra, which we omit, the characteristic equation

$$(\eta - \mu) F(\eta, \mu) = \eta \Delta(\eta) B(\eta) \tag{4}$$

$$B(\eta) = \frac{1}{\sqrt{\pi}} \int\limits_{\infty}^{\infty} \exp(-\mu^2) F(\eta, \mu) d\mu, \quad \Delta(\eta) = \frac{1}{\eta^2 - 3} \begin{Vmatrix} -3 & -\gamma(\eta^2 - {}^3/_2) \\ 1/\gamma & \eta^2 - {}^5/_2 \end{Vmatrix} \tag{5}$$

($B(\eta)$ is a normalizing non-singular vector).

From (4) we find the generalized eigenvectors of a continuous spectrum which fills the entire real line

$$F(\eta, \mu) = [\eta P(\eta - \mu)^{-1} + \omega(\eta) \delta(\eta - \mu)] \Delta(\eta) B(\eta) \tag{6}$$

The symbol Px^{-1} denotes the Cauchy integral's principal value distribution, $\delta(x)$ is the Dirac delta function, and $\omega(\eta)$ is a scalar function determined by the normalizing conditions (5).

Substituting (6) into the first relation in (5), we obtain the equation

$$[\Lambda(\eta) - \exp(-\eta^2) \omega(\eta) \Delta(\eta)] B(\eta) = 0 \tag{7}$$

$$\Lambda(z) = \sqrt{\pi} I + z t(z) \Delta(z), \quad t(z) = \int\limits_{-\infty}^{\infty} \frac{\exp(-x^2)}{x - z} dx$$

where $\Lambda(z)$ is the dispersion matrix and I is the second-order unit matrix.

Eq.(7) shows that the determinant of the matrix expression in square brackets in (7) is equal to zero, which gives a quadratic equation for $\omega(\eta)$ with two solutions $\omega_1(\eta)$ and $\omega_2(\eta)$. Thus formula (6) contains two eigenvectors, which with the help of (7) can be written in the form

$$F_\alpha(\eta, \mu) = [\eta P(\eta - \mu)^{-1} I + \exp(\eta^2) \Omega(\eta) \delta(\eta - \mu)] M_\alpha(\eta), \quad \alpha = 1, 2 \tag{8}$$

$$\Omega(z) = \Lambda(z) \Delta^{-1}(z), \quad M_\alpha(\eta) = \Delta(\eta) B_\alpha(\eta) \tag{9}$$

where $B_\alpha(\eta)$ is a normalizing vector given by the first formula in (5) with the vector $F_\alpha(\eta, \mu)$ substituted into the right-hand side.

We now introduce the dispersion function $\lambda(z) = \det \Lambda(z)$, using which one can verify that the discrete spectrum of the characteristic equation contains the double point $\eta_i = \infty$ to which there correspond two solutions of (1):

$$Z_+(x, \mu) = \begin{Vmatrix} a_1 \\ a_2 \end{Vmatrix}, \quad Z_-(x, \mu) = (\mu - x) Z_+(x, \mu) \tag{10}$$

where a_1 and a_2 are aribitrary constants, and, in view of the second boundary condition (2), only the first vector in (10) participates in the expansion of the solution of (1) in eigenvectors of the characteristic Eq.(4).

Theorem 1. Eq.(1) with boundary conditions (2) has a unique solution which can be represented in the form of the expansion

$$Z(x, \mu) = \varepsilon_T \begin{Vmatrix} -1 \\ 1/\gamma \end{Vmatrix} + \sum_{\alpha=1}^{2} \int\limits_0^\infty \exp\left(-\frac{x}{\eta}\right) a_\alpha(\eta) F_\alpha(\eta, \mu) d\eta \tag{11}$$

i.e. the scalar coefficients ε_T and $a_\alpha(\eta)$ $(\alpha = 1, 2)$ of this expansion are uniquely defined.

Theorem 1 shows that the eigenvectors (8) and (10) form a complete family or basis.

Proof. In accordance with boundary conditions (2) we obtain from expansion (11) the integral equation

$$A\mu \begin{Vmatrix} -1 \\ 1/\gamma \end{Vmatrix} + \varepsilon_n \begin{Vmatrix} 1 \\ 0 \end{Vmatrix} = \varepsilon_T \begin{Vmatrix} -1 \\ 1/\gamma \end{Vmatrix} + \sum_{\alpha=1}^{2} \int_0^\infty a_\alpha(\eta) F_\alpha(\eta, \mu)\, d\eta \tag{12}$$

To prove the theorem we shall demonstrate the existence and uniqueness of the expansion coefficients in (12), and at the same time explicitly compute the temperature jump coefficient ε_T.

We substitute the eigenvectors (8) into expansion (12). We obtain a singular vector integral equation with Cauchy kernel

$$\Psi(\mu) = \int_0^\infty \frac{\eta A(\eta)}{\eta - \mu}\, d\eta + \exp(\mu^2)\,\Omega(\mu)\, A(\mu) \tag{13}$$

where we have put

$$\Psi(\mu) = (A\mu - \varepsilon_T) \begin{Vmatrix} -1 \\ 1/\gamma \end{Vmatrix} + \varepsilon_n \begin{Vmatrix} 1 \\ 0 \end{Vmatrix}$$

and have introduced the new unknown vector

$$A(\eta) = \sum_{\alpha=1}^{2} a_\alpha(\eta) M_\alpha(\eta) \tag{14}$$

We introduce an auxiliary vector function

$$N(z) = \frac{1}{2\pi i} \int_0^\infty \frac{\eta A(\eta)}{\eta - z}\, d\eta \tag{15}$$

analytic in the complex plane with a cut along $\mathbb{R}_+$. Using boundary values above and below $\mathbb{R}_+$ for the matrix (9) and the vector (15), we reduce Eq.(13) to a matrix Riemann-Hilbert boundary-value problem with matrix coefficients:

$$\Omega^+(\mu)\,(2\pi i N^+(\mu) - \Psi(\mu)) = \Omega^-(\mu)\,(2\pi i N^-(\mu) - \Psi(\mu)) \tag{16}$$

Both here and below $\mu \in \mathbb{R}_+$.

We will construct a fundamental matrix function for this problem, i.e. a matrix $X(z)$ that is analytic and non-degenerate in the plane $\mathbb{C}$ with cut $\mathbb{R}_+$ such that the coefficients of the problem are factorized on the edges of this cut:

$$\Omega^+(\mu)\, X^+(\mu) = \Omega^-(\mu)\, X^-(\mu) \tag{17}$$

We will look for a fundamental matrix in the form of the product

$$X(z) = S(z)\, U^{-1}(z)\, S^{-1}(z), \quad U(z) = \operatorname{diag}\{U_1(z), U_2(z)\} \tag{18}$$

where $U(z)$ is a new unknown diagonal matrix and $S(z)$ is a matrix which reduces the matrix $\Omega(z)$ to diagonal form. Direct calculation shows that such a matrix exists:

$$S(z) = \begin{Vmatrix} z^2 + 1/2 + R(z) & z^2 + 1/2 - R(z) \\ -3\gamma & -3\gamma \end{Vmatrix}$$

$$R(z) = \sqrt{w(z)}, \quad w(z) = z^4 - 3z^2 + 25/4$$

We will consider the matrix $S(z)$ to be a single-valued analytic matrix function in the plane with cut $\Gamma = [-\bar{a}, a] \cup [-a, \bar{a}]$, where $\pm a$ and $\pm\bar{a}$ are zeros of the polynomial $w(z)$ with $a = \sqrt{2} + i/\sqrt{2}$. Computing the elements of the diagonal matrix

$$D(z) = S^{-1}(z)\,\Omega(z)\, S(z) = \operatorname{diag}\{D_1(z), D_2(z)\}$$

we obtain

$$D_\alpha(z) = zt(z) + 1/4\sqrt{\pi}\,(11/2 - z^2 \mp R(z)) \quad (\alpha = 1, 2)$$

where both here and below $\alpha = 1$ corresponds to the upper sign and $\alpha = 2$ to the lower.

The factorization problem (18) is now equivalent to a system of two matrix boundary-value problems, one of which is considered on the basic cut:

$$D^+(\mu)\,[U^+(\mu)]^{-1} = D^-(\mu)\,[U^-(\mu)]^{-1} \tag{19}$$

and the other on the additional cut:

$$U^+(\tau)\, T = T U^-(\tau) \quad (\tau \in \Gamma) \tag{20}$$

$$T = [S^+(\tau)]^{-1} S^-(\tau) = \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix}$$

We will write problems (19) and (20) in the form of two systems of boundary-value problems:

$$U_\alpha^+(\mu) = \frac{D_\alpha^+(\mu)}{D_\alpha^-(\mu)} U_\alpha^-(\mu) \quad (\alpha = 1, 2) \tag{21}$$

$$U_1^\pm(\tau) = U_2^\mp(\tau) \quad (\tau \in \Gamma) \tag{22}$$

Let $\theta_\alpha(\mu)$ be the principal value of the argument of the function

$$D_\alpha^+(\mu) = D_\alpha(\mu) + \pi i \mu \exp(-\mu^2)$$

It is clear that

$$\theta_\alpha(\mu) = \begin{cases} \theta_\alpha^{(0)}(\mu), & 0 \leqslant \mu \leqslant x_\alpha \\ \pi + \theta_\alpha^{(0)}(\mu), & x_\alpha < \mu < \infty \end{cases}$$

$$\theta_\alpha^{(0)}(x) = \operatorname{arctg}(\pi x \exp(-x^2)/D_\alpha(x))$$

where x_α is a zero of the function $D_\alpha(x)$.

In order to obtain the solution $\{U_1, U_2\}$ of problems (21) and (22) considered on the basic and additional cuts, we perform obvious transformations on problems (21) and (22) to obtain the following two boundary-value problems, defined only on the basic cut:

$$\ln(U_1U_2)^+ - \ln(U_1U_2)^- = 2ia(\mu) \tag{23}$$

$$\frac{1}{R(\mu)} \ln\left(\frac{U_1}{U_2}\right)^+ - \frac{1}{R(\mu)} \ln\left(\frac{U_1}{U_2}\right)^- = 2i\, \frac{b(\mu)}{R(\mu)} \tag{24}$$

$$a(x) = \theta_1(x) + \theta_2(x) - 2\pi$$

$$b(x) = \theta_1(x) - \theta_2(x)$$

The system of boundary-value problems (23) and (24) is solved by standard methods and its solution has the form

$$U_\alpha^{(*)}(z) = \exp[A(z) \pm R(z) B(z)] \quad (\alpha = 1, 2) \tag{25}$$

$$A(z) = \frac{1}{2\pi} \int_0^\infty \frac{a(\tau)\, d\tau}{\tau - z}, \quad B(z) = \frac{1}{2\pi} \int_0^\infty \frac{b(\tau)\, d\tau}{R(\tau)(\tau - z)}$$

This solution's drawback is the presence of an essential singularity at infinity. In order to cancel the singularity we look for $U_\alpha(z)$ in the form

$$U_1(z) = U_1^{(*)}(z)\, \varphi(z), \quad U_2(z) = U_2^{(*)}(z)/\varphi(z) \tag{26}$$

where $\varphi(z)$ is a function analytic throughout Γ (with an essential singularity at infinity). Here the boundary condition (21) is automatically satisfied, and the boundary condition (22) will be satisfied if and only if

$$\varphi^+(\tau) = 1/\varphi^-(\tau)$$

We take $\varphi(z)$ in the form

$$\varphi(z) = \exp\left(-R(z) \int_0^{\mu_0} \frac{d\mu}{R(\mu)(\mu - z)}\right)$$

For all $\mu_0 \in \mathbb{R}_+$ this function $\varphi(z)$ satisfies the conditions given above, so that the pair $\{U_1, U_2\}$ will be a solution of problems (21) and (22).

In order for the functions $U_\alpha(z)$ $(\alpha = 1, 2)$ not to contain essential singularities, it is necessary and sufficient that

$$\frac{1}{2\pi}\int_0^\infty \frac{b(x)\,dx}{R(x)} = \int_0^{\mu_0} \frac{dx}{R(x)} \tag{27}$$

Problem (27) is a special case of the Jacobi inversion problem.

We note that the function $U_1(z)$ has a second-order zero at the point $z = 0$ and a first-order pole at the point μ_0. The function $U_2(z)$ has a first-order zero at μ_0. We remark that the matrix $X(z)$ is analytic at the zeros of the polynomial $w(z)$ and $\det X(z) \neq 0$, $z \in \mathbb{C} \setminus \mathbb{R}_+$. Thus the fundamental matrix $X(z)$ has been constructed.

We return to problem (16), which with the help of (17) is transformed into a homogeneous vector boundary-value problem with matrix coefficients

$$[X^+(\mu)]^{-1}(2\pi i N^+(\mu) - \Psi(\mu)) = [X^-(\mu)]^{-1}(2\pi i N^-(\mu) - \Psi(\mu)) \tag{28}$$

Taking into account the behaviour at infinity and at the point μ_0 of the vectors and matrices in (28), we obtain the general solution of this problem

$$2\pi i N(z) = (Az - \varepsilon_T)\begin{Vmatrix} -1 \\ 1/\gamma \end{Vmatrix} + \varepsilon_n \begin{Vmatrix} 1 \\ 0 \end{Vmatrix} + X(z)\begin{Vmatrix} \alpha_1 z + \alpha_0 + \dfrac{\alpha_{-1}}{z-\mu_0} \\ \beta_1 z + \beta_0 + \dfrac{\beta_{-1}}{z-\mu_0} \end{Vmatrix} \tag{29}$$

where α_i and β_i $(i = -1, 0, 1)$ are constants. To make this solution well-behaved, i.e. to give this vector the same behaviour at infinity as the vector (15), and to ensure the absence of singularities at finite points, we equate the coefficients of z and z^0 in the expansions of the column elements in (29) to zero, and require that $N(z)$ should not have double poles at the points $z = 0$ and $z = \mu_0$. We thus obtain α_i, β_i, ε_n and ε_T, with

$$\varepsilon_T = -\frac{3lk}{\sqrt{\pi}}\left\{\frac{1}{2\pi}\int_0^\infty \left[a(x) - x^2\frac{b(x)}{R(x)}\right]dx + \int_0^{\mu_0}\frac{x^2\,dx}{R(x)} + \mu_0 + 2\mu_0\frac{\gamma^2+\gamma\beta\mu_0-\alpha^2}{(\gamma-\alpha)^2}\right\} \tag{30}$$

$$\alpha = -{}^1/_2\gamma\,[R(\mu_0) + \mu_0^2 + {}^1/_2],\quad \beta = -\gamma\mu_0\,(1 + (\mu_0^2 - {}^3/_2)/R(\mu_0))$$

Because of the way the vector $N(z)$ was constructed, the coefficients $a_\alpha(\mu)$ are uniquely defined by Sokhotskii's formula $N^+(\mu) - N^-(\mu) = \mu A(\mu)$.

Thus Theorem 1 has been completely proved. It gives the solution of the integrodifferential Eq.(1) in the form (11), describing the temperature jump in a rarefied gas. Eq.(3) gives the required magnitude of the temperature jump.

We will make some comments on the method of solution and the difficulties which had to be overcome to solve (1). The point is that methods of solving a vector boundary-value problem with matrix coefficients where the problem's matrix diagonalizing coefficient $S(z)$ has branch points, have not yet been described in the textbooks. Furthermore, considerable analytic efforts are required here in order to obtain the solution of the matrix boundary-value problems (19) and (20), defined on the basic and two additional cuts. The solution of the basic and supplementary boundary-value problems contains an essential singularity at infinity which is removed using problem (27), which is a special case of the Jacobi inversion problem.

The method presented here can be used to solve various boundary-value problems in kinetic theory, where the Boltzmann equation is used with BGK collision operators.

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